A TEST COMPLEX FOR GORENSTEINNESS

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Abstract. Let R be a commutative noetherian ring with a dualizing complex. By recent work of Iyengar and Krause [9], the difference between the category of acyclic complexes and its subcategory of totally acyclic complexes measures how far R is from being Gorenstein. In particular, R is Gorenstein if and only if every acyclic complex is totally acyclic.

In this note we exhibit a specific acyclic complex with the property that it is totally acyclic if and only if R is Gorenstein.

Introduction

Let R be a commutative noetherian ring. A complex X of R-modules is said to be *acyclic* if it has zero homology, i.e. $\mathrm{H}(X) = 0$. An acyclic complex of projective modules is called *totally acyclic* if the acyclicity is preserved by $\mathrm{Hom}_R(-, P)$ for every projective module P. Dually, an acyclic complex of injective modules is totally acyclic if the acyclicity is preserved by $\mathrm{Hom}_R(I, -)$ for every injective module I.

Over a Gorenstein ring, every acyclic complex of projective or of injective modules is totally acyclic. Iyengar and Krause have recently proved a converse; indeed, by [9, cor. 5.5] the following are equivalent when R has a dualizing complex:

(i) The ring R is Gorenstein.

(ii) Every acyclic complex of projective R-modules is totally acyclic.

(iii) Every acyclic complex of injective R-modules is totally acyclic.

Moreover, for a local ring $(R, \mathfrak{m})$ that is not Gorenstein and has $\mathfrak{m}^2 = 0$ there is a natural example, provided by [9, prop. 6.1(3)], of an acyclic, but not totally acyclic, complex of projective R-modules.

The purpose of this note is to prove that for every ring R with a dualizing complex D, a specific acyclic complex K, defined in 2.1, serves as a test complex for Gorensteinness in the following sense: The ring R is Gorenstein if and only if $K \otimes_R D$ is acyclic. This is achieved by Theorem 2.2. In general, K is an acyclic complex of flat R-modules. Corollary 2.6 shows that if R is an artinian local ring, then K is a complex of projective modules, and (i)–(iii) above are equivalent with

(iv) The complex K is totally acyclic.

Test complexes of injective modules can be obtained directly from K (Corollary 2.5) or through a potentially different construction explored in Section 3. The authors of [9] have pointed out that the latter is of particular interest, as it yields a generator for $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj\ R}) \,/\, \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj\ R})$, the Verdier quotient of acyclic complexes modulo totally acyclic complexes in the homotopy category of injective R-modules. This is proved in Theorem 3.5.

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1. Background

Throughout this paper R is a commutative noetherian ring. The notation $(R,\mathfrak{m},k)$ means R is local with maximal ideal $\mathfrak{m}$ and residue field k.

Complexes of R-modules (R-*complexes* for short) are graded homologically,

$$X = \cdots \to X_{i+1} \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \to \cdots .$$

The *suspension* of X is denoted ΣX; it is the complex with $(\Sigma X)_i = X_{i-1}$ and differential $\partial^{\Sigma X} = -\partial^X$. A complex X is said to be *bounded* if $X_i = 0$ for $|i| \gg 0$.

An isomorphism between R-complexes is denoted by a '$\cong$'; we write $X \cong Y$ if there exists an isomorphism $X \xrightarrow{\cong} Y$.

A morphism between R-complexes is called a *quasi-isomorphism*, and denoted $X \xrightarrow{\simeq} Y$ if the induced map in homology, $\mathrm{H}(X) \to \mathrm{H}(Y)$, is an isomorphism. Following [1, sec. 1] we write $X \simeq Y$, if X and Y can be linked by a sequence of quasi-isomorphisms with arrows in alternating directions. Recall that a morphism $X \to Y$ is a quasi-isomorphism if and only if its mapping cone, written $\mathrm{Cone}\,(X \to Y)$, is acyclic.

1.1. Resolutions. The following facts are established in [1, sec. 1][1] and [2].

Every R-complex X has a semi-projective resolution. That is, there is a quasi-isomorphism $P \xrightarrow{\simeq} X$, where P is a complex of projective R-modules such that $\mathrm{Hom}_R(P,-)$ preserves quasi-isomorphisms. For such a complex, also the functor $- \otimes_R P$ preserves quasi-isomorphisms. In particular, for any R-complexes $Y \simeq Z$ we have $\mathrm{Hom}_R(P,Y) \simeq \mathrm{Hom}_R(P,Z)$ and $Y \otimes_R P \simeq Z \otimes_R P$.

If there is an l such that $\mathrm{H}_i(X) = 0$ for $i < l$, then X has a semi-projective resolution P with $P_i = 0$ for $i < l$. If, in addition, $\mathrm{H}_i(X)$ is finitely generated for all i, then P can be chosen with all modules P_i finitely generated.

Every R-complex X has a semi-injective resolution. That is, there is a quasi-isomorphism $X \xrightarrow{\simeq} J$, where J is a complex of injective R-modules such that $\mathrm{Hom}_R(-,J)$ preserves quasi-isomorphisms. In particular, for such a complex J and any R-complexes $Y \simeq Z$ we have $\mathrm{Hom}_R(Y,J) \simeq \mathrm{Hom}_R(Z,J)$.

1.2. Lemma. *Let X and Y be R-complexes such that either $X_i = 0$ for all $i \ll 0$ or $Y_i = 0$ for all $i \gg 0$. If $\mathrm{H}(X_i \otimes_R Y) = 0$ for all $i \in \mathbb{Z}$, then $\mathrm{H}(X \otimes_R Y) = 0$.*

Proof. Let E be a faithfully injective R-module. The complex $X \otimes_R Y$ is acyclic if and only if $\mathrm{Hom}_R(X \otimes_R Y, E) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y,E))$ is so. The claim is now immediate from [5, lem. (2.4)]. □

1.3. Dualizing complexes. Following [8, V.§2], a *dualizing complex* for R is a bounded complex D of injective R-modules such that $\mathrm{H}_i(D)$ is finitely generated for all $i \in \mathbb{Z}$, and the homothety morphism

$$\chi^D \colon R \longrightarrow \mathrm{Hom}_R(D,D)$$

is a quasi-isomorphism.

Let $(R,\mathfrak{m},k)$ be a local ring with a dualizing complex D. After suspensions we can assume D is *normalized*, cf. [8, V.§5], in which case [8, prop. V.3.4] yields

$$\mathrm{H}(\mathrm{Hom}_R(k,D)) \cong k. \tag{1.3.1}$$

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[1] Where semi-projective/injective resolutions are called DG-projective/injective.

If R is artinian, then $\mathrm{E}_R(k)$, the injective hull of the residue field, is a normalized dualizing complex for R.

2. A test complex of flat modules

2.1. A distinguished complex of flat modules. Assume that R has a dualizing complex D, and let $\pi\colon P \xrightarrow{\simeq} D$ be a semi-projective resolution. By 1.1 we can assume that P consists of finitely generated modules with $P_i = 0$ for all $i \ll 0$. The functors $\mathrm{Hom}_R(P,-)$ and $\mathrm{Hom}_R(-,D)$ preserve quasi-isomorphisms, so the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(P,P) \arrow[r, "\mathrm{Hom}_R(P,\pi)", "\simeq"'] & \mathrm{Hom}_R(P,D) \\
R \arrow[u, "\chi^P"] \arrow[r, "\chi^D", "\simeq"'] & \mathrm{Hom}_R(D,D) \arrow[u, "\simeq", "\mathrm{Hom}_R(\pi,D)"']
\end{tikzcd}$$

shows that also the homothety map χ^P is a quasi-isomorphism. In particular,

$$K = \mathrm{Cone}\,(R \xrightarrow{\chi^P} \mathrm{Hom}_R(P,P))$$

is acyclic. The modules in $\mathrm{Hom}_R(P,P)$ are direct products of modules of the form $\mathrm{Hom}_R(P_i,P_{i+n})$, and each such module is flat. Thus, χ^P is a quasi-isomorphism between complexes of flat R-modules, and the mapping cone K is, therefore, an acyclic complex of flat R-modules.

We can now state the main result; the proof is given at the end of the section.

2.2. Theorem. *Let R be a commutative noetherian ring with a dualizing complex D, and let K be the acyclic complex of flat modules defined in 2.1. The ring R is Gorenstein if and only if the complex $K \otimes_R D$ is acyclic.*

2.3. Remark. While also $C = \mathrm{Cone}\,\chi^D$ is an acyclic complex of flat R-modules, it cannot detect Gorensteinness. Indeed, C is bounded, so $C \otimes_R X$ is acyclic for every R-complex X by Lemma 1.2. If R is artinian, then C is even split exact.

2.4. Remark. In the theory of Gorenstein dimensions, there is a notion of a *complete flat resolution*—due to Enochs, Jenda, and Torrecillas [6]—namely an acyclic complex F of flat modules such that $F \otimes_R I$ is acyclic for every injective module I.

If R is Gorenstein, then every acyclic complex of flat R-modules is a complete flat resolution. Indeed, every injective R-module I has finite flat dimension, and then it is straightforward to verify that the functor $- \otimes_R I$ preserves acyclicity of complexes of flat modules. On the other hand, let K and D be as in Theorem 2.2. If K is a complete flat resolution, then $K \otimes_R D$ is acyclic by Lemma 1.2.

Thus, the following assertions are equivalent:

(*i*) The ring R is Gorenstein.

(*ii*) The complex K is a complete flat resolution.

(*iii*) Every acyclic complex of flat modules is a complete flat resolution.

The complex K defined in 2.1 appears to be a natural test object for Gorensteinness. However, it might in the context of [9] be of interest to exhibit a test complex of injective or of projective modules.

To this end, we first note that the next corollary to Theorem 2.2 is immediate in view of Remark 2.4 and [3, prop. (6.4.1)]. See Section 3 for a further discussion of test complexes of injective modules.

2.5. Corollary. *Let R be a commutative noetherian ring with a dualizing complex. Let K be the acyclic complex of flat modules defined in 2.1, and let E be a faithfully injective R-module. The complex $\operatorname{Hom}_R(K,E)$ is an acyclic complex of injective modules, and R is Gorenstein if and only if $\operatorname{Hom}_R(K,E)$ is totally acyclic.* □

For artinian local rings $(R,\mathfrak{m})$, Theorem 2.2 provides a test complex of projective modules. In particular, for R with $\mathfrak{m}^2=0$ the following recovers [9, prop. 6.1(3)].

2.6. Corollary. *Let R be an artinian local ring. The complex K defined in 2.1 is an acyclic complex of projective R-modules, and R is Gorenstein if and only if K is totally acyclic.*

Proof. When R is artinian and local, every flat R-module is projective. Thus, K is an acyclic complex of projective modules.

The "only if" part is well-known. To prove "if", assume K is totally acyclic and recall from 1.3 that the module $E=\mathrm{E}_R(k)$ is dualizing for R. The first of the following isomorphisms is induced by χ^E, and the second is Hom-tensor adjointness

$$\operatorname{Hom}_R(K,R)\cong\operatorname{Hom}_R(K,\operatorname{Hom}_R(E,E))\cong\operatorname{Hom}_R(K\otimes_R E,E).$$

The complex $\operatorname{Hom}_R(K,R)$ is acyclic and E is faithfully injective, so $K\otimes_R E$ is acyclic and, therefore, R is Gorenstein by Theorem 2.2. □

For the proof of Theorem 2.2 we need a technical lemma.

2.7. Lemma. *Let P be an R-complex of finitely generated projective modules, X be any R-complex, and B be a bounded R-complex of finitely generated modules. There is an isomorphism of R-complexes*

$$\omega\colon \operatorname{Hom}_R(P,X)\otimes_R B \xrightarrow{\cong} \operatorname{Hom}_R(P,X\otimes_R B).$$

Proof. It is straightforward to check that the assignment

$$\omega(\phi\otimes b)(p)=(-1)^{|p||b|}\phi(p)\otimes b,$$

where $|\cdot|$ denotes the degree of an element, defines a morphism between the relevant complexes. By assumption, there exist integers $l\leqslant u$ such that $B_h=0$ when $h<l$ or $h>u$. For every $n\in\mathbb{Z}$ we have

$$\begin{aligned}
(\operatorname{Hom}_R(P,X)\otimes_R B)_n &= \bigoplus_{i=n-u}^{n-l}\operatorname{Hom}_R(P,X)_i\otimes_R B_{n-i}\\
&= \bigoplus_{i=n-u}^{n-l}\big(\prod_{j\in\mathbb{Z}}\operatorname{Hom}_R(P_j,X_{j+i})\big)\otimes_R B_{n-i}\\
&\cong \bigoplus_{i=n-u}^{n-l}\prod_{j\in\mathbb{Z}}(\operatorname{Hom}_R(P_j,X_{j+i})\otimes_R B_{n-i})\\
&\cong \bigoplus_{i=n-u}^{n-l}\prod_{j\in\mathbb{Z}}\operatorname{Hom}_R(P_j,X_{j+i}\otimes_R B_{n-i})
\end{aligned}$$

$$
\begin{aligned}
&\cong \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(P_j, \bigoplus_{i=n-u}^{n-l} X_{j+i} \otimes_R B_{n-i}) \\
&= \prod_{j\in\mathbb{Z}} \operatorname{Hom}_R(P_j, (X \otimes_R B)_{j+n}) \\
&= \operatorname{Hom}_R(P, X \otimes_R B)_n.
\end{aligned}
$$

Since the modules B_{n-i} are finitely generated, the functors $- \otimes_R B_{n-i}$ commute with arbitrary products for every i; this explains the first isomorphism. The modules P_j are finitely generated and projective, so for all i, j, and n the homomorphism of modules

$$\operatorname{Hom}_R(P_j, X_{j+i}) \otimes_R B_{n-i} \xrightarrow{\omega_{ijn}} \operatorname{Hom}_R(P_j, X_{j+i} \otimes_R B_{n-i})$$

is invertible, and this accounts for the second isomorphism. Thus, ω is an isomorphism of graded modules, and the sign in the definition of ω ensures that it commutes with the differentials. $\square$

Proof of Theorem 2.2. The "only if" part was settled in Remark 2.4.

For the "if" part, assume that the complex $K \otimes_R D$ is acyclic; the isomorphism $\operatorname{Cone}(\chi^P \otimes_R D) \cong K \otimes_R D$ implies that

$$\chi^P \otimes_R D\colon D \longrightarrow \operatorname{Hom}_R(P,P) \otimes_R D \tag{1}$$

is a quasi-isomorphism.

Choose an n such that $\mathrm{H}_i(D) = 0$ for all $i > n$, and let B be the soft truncation of P on the left at n:

$$B = 0 \longrightarrow \operatorname{Coker} \partial_{n+1}^P \xrightarrow{\overline{\partial_n^P}} P_{n-1} \xrightarrow{\partial_{n-1}^P} P_{n-2} \longrightarrow \cdots .$$

There are quasi-isomorphisms $B \xleftarrow{\simeq} P \xrightarrow{\simeq} D$ and, hence, a quasi-isomorphism $\beta\colon B \xrightarrow{\simeq} D$; see [1, 1.1.I.(1) and 1.4.I]. Since the mapping cone of β is a bounded acyclic complex, and $\operatorname{Hom}_R(P,P)$ is a complex of flat modules, Lemma 1.2 applies to show that also $\operatorname{Hom}_R(P,P) \otimes_R \operatorname{Cone}(\beta)$ is acyclic. Thus, the isomorphism $\operatorname{Cone}(\operatorname{Hom}_R(P,P) \otimes_R \beta) \cong \operatorname{Hom}_R(P,P) \otimes_R \operatorname{Cone}(\beta)$ implies that also

$$\operatorname{Hom}_R(P,P) \otimes_R \beta\colon \operatorname{Hom}_R(P,P) \otimes_R B \longrightarrow \operatorname{Hom}_R(P,P) \otimes_R D \tag{2}$$

is a quasi-isomorphism.

By the choice of P, cf. 2.1, the bounded complex B consists of finitely generated modules, and Lemma 2.7 yields an isomorphism

$$\omega\colon \operatorname{Hom}_R(P,P) \otimes_R B \xrightarrow{\cong} \operatorname{Hom}_R(P, P \otimes_R B). \tag{3}$$

Finally, let $\iota\colon P \otimes_R B \xrightarrow{\simeq} J$ be a semi-injective resolution; the quasi-isomorphism ι is preserved by $\operatorname{Hom}_R(P,-)$, and the resulting quasi-isomorphism combined with (1), (2), and (3) yields

$$D \simeq \operatorname{Hom}_R(P,J). \tag{4}$$

It suffices to prove that $R_{\mathfrak{m}}$ is Gorenstein for every maximal ideal $\mathfrak{m}$ of R. Let $\mathfrak{m}$ be a maximal ideal; the complex $D_{\mathfrak{m}}$ is dualizing for $R_{\mathfrak{m}}$, see [8, cor. V.2.3]. Set $k = R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} \cong R/\mathfrak{m}$. We may, after suspensions, assume $D_{\mathfrak{m}}$ is normalized, so $\mathrm{H}(\operatorname{Hom}_{R_{\mathfrak{m}}}(k, D_{\mathfrak{m}})) \cong k$; see (1.3.1). Moreover, there are isomorphisms $\operatorname{Hom}_R(k,D) \cong \operatorname{Hom}_R(k,D) \otimes_R R_{\mathfrak{m}} \cong \operatorname{Hom}_{R_{\mathfrak{m}}}(k, D_{\mathfrak{m}})$, so we have

$$k \cong \mathrm{H}(\operatorname{Hom}_R(k,D)). \tag{5}$$

Let $\upsilon\colon Q \xrightarrow{\simeq} k$ be a semi-projective resolution of k over R. As $\operatorname{Hom}_R(-,D)$ preserves quasi-isomorphisms, we have

$$\operatorname{Hom}_R(k,D) \xrightarrow{\simeq} \operatorname{Hom}_R(Q,D). \tag{6}$$

Also $\operatorname{Hom}_R(Q,-)$ preserves quasi-isomorphisms, and from (4) we get

$$\operatorname{Hom}_R(Q,D) \simeq \operatorname{Hom}_R(Q,\operatorname{Hom}_R(P,J)) \cong \operatorname{Hom}_R(Q\otimes_R P,J), \tag{7}$$

where the isomorphism is Hom-tensor adjointness. Finally, $\upsilon\otimes_R P$ is a quasi-isomorphism, and hence so is

$$\operatorname{Hom}_R(\upsilon\otimes_R P,J)\colon \operatorname{Hom}_R(k\otimes_R P,J) \xrightarrow{\simeq} \operatorname{Hom}_R(Q\otimes_R P,J). \tag{8}$$

Combining (5)–(8) and again using Hom-tensor adjointness, we obtain

$$\begin{aligned} k &\cong \mathrm{H}(\operatorname{Hom}_R(k\otimes_R P,J)) \\ &\cong \mathrm{H}(\operatorname{Hom}_R((k\otimes_R P)\otimes_k k,J)) \\ &\cong \mathrm{H}(\operatorname{Hom}_k(k\otimes_R P,\operatorname{Hom}_R(k,J))) \\ &\cong \operatorname{Hom}_k(\mathrm{H}(k\otimes_R P),\mathrm{H}(\operatorname{Hom}_R(k,J))). \end{aligned}$$

Thus, $\operatorname{Hom}_k(\mathrm{H}(k\otimes_R P),\mathrm{H}(\operatorname{Hom}_R(k,J)))$ is a finitely generated k-vector space; in particular, $\mathrm{H}(k\otimes_R P)$ must be finitely generated. Note that $\mathrm{H}_i(k\otimes_R P)\cong \mathrm{H}_i(k\otimes_{R_\mathfrak{m}} P_\mathfrak{m})$ for all $i\in\mathbb{Z}$; it follows that $\mathrm{H}_i(k\otimes_{R_\mathfrak{m}} P_\mathfrak{m})=0$ for all $i\gg 0$. By [1, prop. 5.5] the dualizing $R_\mathfrak{m}$-complex $D_\mathfrak{m}$ then has finite flat dimension, and hence $R_\mathfrak{m}$ is Gorenstein; see [7, thm. (17.23)] or [4, thm. (8.1)]. □

3. A test complex of injective modules

The next construction is another source for test complexes.

3.1. A distinguished complex of injective modules. Assume R has a dualizing complex D. As in 2.1, let $\pi\colon P \xrightarrow{\simeq} D$ be a semi-projective resolution of D consisting of finitely generated modules with $P_i=0$ for all $i\ll 0$. The assignment $\varphi\otimes p\mapsto\varphi(p)$ defines a morphism of complexes, ε, such that the following diagram is commutative:

$$\begin{tikzcd} \operatorname{Hom}_R(P,D)\otimes_R P \arrow[rr, "\varepsilon"] & & D \\ \operatorname{Hom}_R(P,P)\otimes_R P \arrow[u, "\operatorname{Hom}_R(P,\pi)\otimes_R P", "\simeq"'] & & R\otimes_R P. \arrow[ll, "\simeq", "\chi^P\otimes_R P"'] \arrow[u, "\simeq", "\pi"'] \end{tikzcd}$$

Thus, ε is a quasi-isomorphism between complexes of injective R-modules, and the mapping cone

$$M = \operatorname{Cone}(\operatorname{Hom}_R(P,D)\otimes_R P \xrightarrow{\varepsilon} D)$$

an acyclic complex of injective R-modules.

An argument similar to the proof of Theorem 2.2 yields the next result, which is also a corollary of Theorem 3.5.

3.2. Theorem. *Let R be a commutative noetherian ring with a dualizing complex, and let M be the acyclic complex of injective modules defined in 3.1. The ring R is Gorenstein if and only if M is totally acyclic.* □

3.3. Remark. If $(R, \mathfrak{m}, k)$ is an artinian local ring, then there is an isomorphism

$$K \cong \Sigma \mathrm{Hom}_R(M, \mathrm{E}_R(k))$$

where K and M are the complexes from 2.1 and 3.1, and $\mathrm{E}_R(k)$ is the injective hull of k. Indeed, with $E = \mathrm{E}_R(k)$ there is a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(E,E) \arrow[r, "\mathrm{Hom}_R(\varepsilon,E)", "\simeq"'] & \mathrm{Hom}_R(\mathrm{Hom}_R(P,E)\otimes_R P, E) \\
 & \mathrm{Hom}_R(P, \mathrm{Hom}_R(\mathrm{Hom}_R(P,E),E)) \arrow[u, "\cong"] \\
R \arrow[uu, "\cong"', "\chi^E"] \arrow[r, "\chi^P", "\simeq"'] & \mathrm{Hom}_R(P,P). \arrow[u, "\cong"]
\end{tikzcd}$$

The vertical maps on the right are the natural isomorphisms, and because E is a module, also the homothety map χ^E is a genuine isomorphism. The diagram induces the desired isomorphism between the complexes $K = \mathrm{Cone}\,(\chi^P)$ and $\mathrm{Cone}\,(\mathrm{Hom}_R(\varepsilon, E)) \cong \Sigma \mathrm{Hom}_R(\mathrm{Cone}\,(\varepsilon), E) = \Sigma \mathrm{Hom}_R(M, E)$.

When R is not artinian, we do not know if the complexes K and M are related.

Using [5, prop. (5.1)] it is not hard to prove the next parallel to Corollary 2.5.

3.4. Corollary. *Let R be a commutative noetherian ring with a dualizing complex. Let M be the acyclic complex of injective modules defined in 3.1, and let E be a faithfully injective R-module. The complex $\mathrm{Hom}_R(M, E)$ is an acyclic complex of flat modules, and R is Gorenstein if and only if $\mathrm{Hom}_R(M, E) \otimes_R I$ is acyclic for every injective module I.* □

In conversations, the authors of [9] have informed us of Theorem 3.5 below; note that it contains Theorem 3.2. For notation and terminology we refer to [9].

3.5. Theorem. *Let R be a commutative noetherian ring with a dualizing complex. The acyclic complex M of injective modules defined in 3.1 generates the quotient category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj\ R}) \,/\, \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj\ R})$.*

Proof. By [9, 1.7, 5.4, and 5.9(3)] the quotient category $\mathbf{K}_{\mathrm{ac}}(\mathrm{Inj\ R}) \,/\, \mathbf{K}_{\mathrm{tac}}(\mathrm{Inj\ R})$ is generated by the image of the dualizing complex D under the equivalence $\mathbf{D}^f(R) \xrightarrow{\simeq} \mathbf{K}^{\mathrm{c}}(\mathrm{Inj\ R})$, cf. [9, 2.3(2)].

Let $P \xrightarrow{\simeq} D$ be a semi-projective resolution. The functor $\mathrm{Hom}_R(P, -)$ preserves quasi-isomorphisms, so the composite

$$R \xrightarrow{\simeq} \mathrm{Hom}_R(P,P) \xrightarrow{\simeq} \mathrm{Hom}_R(P,D)$$

provides an injective resolution $R \xrightarrow{\simeq} \mathsf{i}R = \mathrm{Hom}_R(P, D)$. Since $\mathsf{i}R$ is a compact object in $\mathbf{K}(\mathrm{Inj\ R})$, the inclusion of the localizing subcategory $\mathrm{Loc}(\mathsf{i}R) \subseteq \mathbf{K}(\mathrm{Inj\ R})$ admits a right adjoint $\rho\colon \mathbf{K}(\mathrm{Inj\ R}) \to \mathrm{Loc}(\mathsf{i}R)$; see [9, 1.5.1]. By [9, 2.3(2)] the image of D in $\mathbf{K}(\mathrm{Inj\ R})$ is

$$\mathrm{Cone}(\rho(D) \xrightarrow{\xi} D),$$

where ξ is the natural map.

It remains to show that $M \cong \mathrm{Cone}(\rho(D) \xrightarrow{\xi} D)$. It suffices to establish a commutative diagram,

$$\begin{tikzcd} \rho(D) \arrow[r, "\xi"] \arrow[d, "\cong"] & D \\ \mathrm{Hom}_R(P,D) \otimes_R P \arrow[ur, "\varepsilon"'] & \end{tikzcd}$$

The complex $\mathrm{Hom}_R(P,D) \otimes_R P = \mathsf{i}R \otimes_R P$ is in $\mathrm{Loc}(\mathsf{i}R)$, and since ε is a quasi-isomorphism, $\mathrm{Hom}_{\mathbf{K}(\mathrm{Inj}\ R)}(\mathsf{i}R, \varepsilon)$ is an isomorphism, cf. [9, 2.2]. The existence of the desired isomorphism $\rho(D) \cong \mathrm{Hom}_R(P,D) \otimes_R P$ now follows from [9, 1.4]. $\square$

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